Mathematical Social Sciences 59 (2010) 148–169

Contents lists available at ScienceDirect

Mathematical Social Sciences

journal homepage: www.elsevier.com/locate/econbase

Sustainability and discounted utilitarianism in models of economic growth[☆]

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ARTICLE INFO

Article history:
Received 29 December 2008
Received in revised form
12 August 2009
Accepted 27 August 2009
Available online 12 September 2009

JEL classification:
D63
D71
Q01

Keywords:
Intergenerational equity
Sustainability
Discounted utilitarianism
Egalitarian consumption streams
Efficiency
Exhaustible resources

ABSTRACT

Discounted utilitarianism treats generations unequally and leads to seemingly unappealing consequences in some models of economic growth. Instead, this paper presents and applies *sustainable discounted utilitarianism* (SDU). SDU respects the interests of future generations and resolves intergenerational conflicts by imposing on discounted utilitarianism that the evaluation be insensitive to the interests of the present generation if the present is better off than the future. An SDU social welfare function always exists. We provide a convenient sufficient condition to identify SDU optima and apply SDU to two well-known models of economic growth.

1. Introduction

Both in the theory of economic growth and in the practical evaluation of economic policy with long-term effects (e.g., climate policies), it is common to apply the *discounted utilitarian* (*DU*) criterion. DU

☆ We thank Wolfgang Buchholz and Bertil Tungodden for the many discussions on this and related projects, two anonymous referees for their valuable suggestions, and Christian Gollier and other seminar participants in Ascona, Milan, Toulouse and Trondheim for their helpful comments. Asheim gratefully acknowledges the hospitality of Cornell University and University of California at Santa Barbara.

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doi:10.1016/j.mathsocsci.2009.08.003

means that one infinite stream of consumption is deemed better than another if and only if it generates a higher sum of utilities discounted by a constant per period discount factor δ, where δ is positive and smaller than one.

In spite of its prevalence, DU is controversial, both due to the conditions through which it is justified and due to its consequences for choice in economically relevant situations. As a matter of principle, DU gives less weight to the utility of future generations and therefore treats generations in an unequal manner. If one abstracts from the probability that the world will be coming to an end, thereby assuming that any generation will appear with certainty, it is natural to question whether it is fair to value the utility of future generations less than that of the present one. This criticism has a long tradition in economics, dating back at least to Pigou (1932).

When applied to some models of economic growth, DU leads to seemingly unappealing consequences. In particular, in the model of capital accumulation and resource depletion first analyzed by Dasgupta and Heal (1974) and Solow (1974) – which we will henceforth refer to as *the Dasgupta–Heal–Solow* (*DHS*) *model* – the application of DU forces consumption to approach zero as time goes to infinity, even though sustainable streams with constant or increasing consumption are feasible. Moreover, this result holds for any discount factor δ smaller than one; even when δ is close to one so that discounting is small. In other words, when applied to the DHS model, the use of DU undermines the livelihood of generations in the far future also when each generation is given almost the same weight as its predecessor.

This motivates the central question posed in this paper: Does there exist an alternative criterion of intergenerational justice satisfying the following desiderata:

(1) The criterion incorporates an equity condition respecting the interests of future generations.
(2) The criterion resolves intergenerational conflicts by leading to consequences with ethical appeal, in particular when applied to the DHS model, as well as to the usual one-sector model of economic growth (*the Ramsey model*).

In our investigation, we adopt a setting that allows for easy comparison with DU, as axiomatized by Koopmans (1960). In particular, we remain within Koopmans' (1960) framework, by requiring our criterion (a) to be representable by a numerical social welfare function (SWF), (b) to satisfy Koopmans' (1960) stationarity condition, and (c) to retain some sensitivity to the interest of the present generation.

One way of ensuring that generations are treated in an equal manner is to insist on the procedural equity condition of *Finite Anonymity*. Finite Anonymity means that a finite permutation of a consumption stream leads to an alternative stream that is equally good in social evaluation. Finite Anonymity has the interesting property that – when combined with the Pareto principle in models of economic growth – it rules out streams that are not non-decreasing, provided that the technology satisfies a productivity condition (see Asheim et al., 2001). Since a DHS technology is productive in this sense, Finite Anonymity combined with the Pareto principle entails that only efficient and non-decreasing streams are acceptable. In particular, it thus formalizes the ethical intuition that deems as unacceptable the consequences of discounted utilitarianism in the setting of DHS technologies.

However, as demonstrated by Basu and Mitra (2003), there exists no numerically representable welfare function which satisfies both Finite Anonymity and the Pareto principle in the setting of infinite streams. This is illustrated by *classical utilitarianism* and *leximin* (adapted to this setting), which satisfy Finite Anonymity and the Pareto principle but are not numerically representable. In fact, Finite Anonymity is hard to combine with any kind of sensitivity to the interests of each generation, as long as one requires numerical representability (see Basu and Mitra, 2007).

An alternative is to apply the axiom of *Hammond Equity for the Future* (*HEF*), which is a weak consequentialist equity condition introduced by Asheim and Tungodden (2004) and analyzed by Asheim et al. (2007, 2009), Banerjee (2006) and Alcantud and García-Sanz (2008). HEF captures the following ethical intuition: a sacrifice by the present generation leading to a uniform gain for all future generations cannot yield a consumption stream that is less desirable in social evaluation if the present remains better off than the future even after the sacrifice. Under consistency requirements on the social preferences, HEF is not only weaker than the ordinary Hammond Equity condition, but it is also implied by other consequentialist equity conditions like the Pigou–Dalton principle of transfers and the Lorenz Domination principle (see Asheim et al., 2007, for details). Hence, it can be endorsed both from an egalitarian and a utilitarian point of view.

Combined with continuity, HEF entails that social evaluation is sensitive to the interests of the present generation only when the present is worse off than the future. As investigated in our companion paper, Asheim et al. (2009), the axiom can be introduced in the Koopmans framework, in which it can be used to justify what we there refer to as a *sustainable recursive SWF*. A sustainable recursive SWF has a continuous and non-decreasing aggregator function over present utility and future welfare, which is increasing in future welfare and exhibits sensitivity for present utility if and only if present utility falls short of future welfare. It is shown in our companion paper that any such SWF satisfies the main Chichilnisky (1996) axioms: *No dictatorship of the present* and *No dictatorship of the future*.

The purpose of the current paper is to apply the concept of sustainable recursive SWFs to two important classes of technologies used to model economic growth: Ramsey technologies and DHS technologies. We thereby demonstrate the applicability of this concept and allow judgements to be made on its consequences in these models. For reasons of tractability, we consider a sub-class of sustainable recursive SWFs, which we refer to as *sustainable discounted utilitarian* (SDU) preferences, obtained by considering a modification of DU preferences consistent with the condition of HEF. The resulting criterion, which we refer to as the SDU criterion, allows for easy comparison with the DU criterion.

SDU avoids the pitfalls of DU (which is too willing to sacrifice future generations), of classical utilitarianism (which is too willing to sacrifice the present generation), and of leximin (which is too willing to ignore possibilities for immense and infinitely lasting future benefits for future generations at a bearable cost to the present generation).

In suggesting an alternative that differs from the three criteria of DU, classical utilitarianism and leximin, SDU follows the lead of Chichilnisky's (1996) *sustainable preference*. However, applied to Ramsey and DHS technologies there exists no optimum under a sustainable preference defined as the sum of a discounted utilitarian part and an asymptotic part (which is an integral with respect to a purely finitely additive measure, cf. Chichilnisky, 1996, Theorem 1 and 2), unless time-variant discounting is used and the discounted utilitarian optimization leads to unbounded consumption growth, making the asymptotic part redundant (cf. Heal, 1998). It is therefore of particular interest to establish that SDU, being an SWF that satisfies the two main Chichilnisky (1996) axioms, is not subject to any similar existence problem in these technological environments.

We now briefly describe the contents of the paper. In Section 2 we present the formal definition of an SDU SWF: an SWF is SDU if it satisfies four requirements. While three of these requirements are also satisfied by DU, one departs from DU by requiring that an SDU SWF not be sensitive to the interests of the present generation if the present is better off than the future. This requirement ensures that an SDU SWF satisfies HEF. We present in this section the important result (Theorem 1) that an SDU SWF always exists, and that it is unique when restricted to the subset of bounded streams. Moreover, we observe (Proposition 1) that any SDU SWF is a sustainable recursive SWF.

In Section 3 we provide a convenient sufficient condition to identify SDU optimum streams within any given set of feasible streams (Proposition 3). In Section 4 we consider the class of Ramsey technologies and characterize the set of SDU optimum streams in this environment (Theorem 2). Likewise, in Section 5 we apply results from earlier work (Dasgupta and Mitra, 1983; Asheim, 1988) and characterize the set of SDU optimum streams in the class of DHS technologies (Theorem 3).

In Section 6 we discuss how SDU resolves distributional conflicts between generations; in particular, in DHS technologies the use of SDU leads to development at first when capital is productive, while protecting the generations in the distant future from the grave consequences of discounting when the vanishing resource stock undermines capital productivity.

The technical parts of the paper, including proofs of all the lemmas, are presented in an Appendix. It is useful for the reader to note that the lemmas whose statements appear in the main text (in Sections 4 and 5) are numbered as Lemmas 1–6, and their proofs are included in Appendix A.5. Lemmas whose statements appear only in the Appendix are preliminary results used to prove the theorems and propositions in the main text and are numbered Lemmas A.1–A.3.

2. Sustainable discounted utilitarian SWFs

Denote by $\mathbb{R}_+$ the set of all non-negative real numbers, by $\mathbb{R}_{++}$ the set of all positive real numbers, by $\mathbb{Z}_+$ the set of all non-negative integers, and by $\mathbb{N}$ the set of all positive integers. Denote by ${}_0\mathbf{x} = (x_0, x_1, \ldots, x_t, \ldots) \in \mathbb{R}_+^{\mathbb{Z}_+}$ an infinite stream of *consumption* where, for $t \in \mathbb{Z}_+$, x_t is a non-negative indicator of the well-being of generation t. Define, for $T \in \mathbb{N}$, ${}_0\mathbf{x}_{T-1} = (x_0, \ldots, x_{T-1})$ and ${}_T\mathbf{x} = (x_T, x_{T+1}, \ldots)$; these are, respectively, the T-head and the T-tail of ${}_0\mathbf{x}$. A consumption stream ${}_0\mathbf{x}$ is called *egalitarian* if $x_t = x_{t+1}$ for all $t \in \mathbb{Z}_+$.

Utility in a period is derived from consumption in that period alone. The *utility function* $U : \mathbb{R}_+ \to \mathbb{R}$ is assumed to satisfy:

$$U \text{ is strictly increasing, strictly concave, and continuous on } \mathbb{R}_+ \tag{U.1}$$

$$U \text{ is continuously differentiable on } \mathbb{R}_{++}, \text{ and } U'(x) \to \infty \text{ as } x \to 0. \tag{U.2}$$

Denote by $\delta \in (0, 1)$ the utility discount factor. Consider the following classes of infinite consumption streams:

$$\mathbf{X}_\delta := \left\{ {}_0\mathbf{x} \in \mathbb{R}_+^{\mathbb{Z}_+} \,\middle|\, \sum_{t=0}^{\infty} \delta^t x_t < \infty \right\}$$

$$\mathbf{X}_\varphi := \left\{ {}_0\mathbf{x} \in \mathbb{R}_+^{\mathbb{Z}_+} \,|_0\, \mathbf{x} \text{ is bounded} \right\}.$$

Note that, if $0 < \delta' < \delta'' < 1$, then $\mathbf{X}_{\delta'} \supsetneq \mathbf{X}_{\delta''} \supsetneq \bigcap_{\delta \in (0,1)} \mathbf{X}_\delta \supsetneq \mathbf{X}_\varphi$.

Given any $\delta \in (0, 1)$, the SWF $w : \mathbf{X}_\delta \to \mathbb{R}$ defined by

$$w({}_0\mathbf{x}) := (1 - \delta) \sum_{t=0}^{\infty} \delta^t U(x_t)$$

is the *discounted utilitarian* (DU) SWF. It follows from (U.1) that w is well-defined. Multiplying the sum of discounted utilities by $1 - \delta$ ensures that $w({}_0\mathbf{x}) = U(x_0)$ if ${}_0\mathbf{x}$ is egalitarian.

The *sustainable discounted utilitarian* (SDU) SWF modifies DU in the following manner. Given any $\delta \in (0, 1)$, an SWF $W : \mathbf{X}_\delta \to \mathbb{R}$ is SDU if

$$W({}_0\mathbf{x}) = \begin{cases} (1-\delta)U(x_0) + \delta W({}_1\mathbf{x}) & \text{if } U(x_0) \le W({}_1\mathbf{x}) \\ W({}_1\mathbf{x}) & \text{if } U(x_0) > W({}_1\mathbf{x}), \end{cases} \tag{W.1}$$

$$W({}_0\mathbf{x}) = U(x_0) \quad \text{if } {}_0\mathbf{x} \text{ is egalitarian}, \tag{W.2}$$

$$W({}_0\mathbf{x}') \ge W({}_0\mathbf{x}'') \quad \text{if } {}_0\mathbf{x}' \ge {}_0\mathbf{x}'', \tag{W.3}$$

$$\lim_{T\to\infty} \delta^{\mathrm{T}} W({}_T\mathbf{x}) = 0. \tag{W.4}$$

Requirement (W.1) departs from DU by requiring that an SDU SWF not be sensitive to the interests of the present generation if the present is better off than the future. In contrast, the other three requirements defining an SDU SWF, (W.2)–(W.4), are also satisfied by DU. They are restrictions which are independent of (W.1).

We now state and prove a result which addresses the issues of existence and uniqueness of a SDU SWF.

Theorem 1. *Let $\delta \in (0, 1)$ and $U : \mathbb{R}_+ \to \mathbb{R}$ satisfying (U.1) be given. Then:*

(i) *There exists an SWF, $\overline{W} : \mathbf{X}_\delta \to \mathbb{R}$ satisfying (W.1)–(W.4);*
(ii) *If $W : \mathbf{X}_\delta \to \mathbb{R}$ is any SDU SWF, then $W({}_0x) = \overline{W}({}_0x)$ for all ${}_0x \in \mathbf{X}_\varphi$;*
(iii) *If $W : \mathbf{X}_\delta \to \mathbb{R}$ is any SDU SWF, then $W({}_0x) \le \overline{W}({}_0x)$ for all ${}_0x \in \mathbf{X}_\delta$.*

Proof. To establish (i), consider the following algorithmic construction. For any stream ${}_0\mathbf{x} \in \mathbf{X}_\delta$ and each $T \in \mathbb{N}$, construct the finite sequence:

$$\left.\begin{array}{l} z(T,T) = w({}_T\mathbf{x}) \\ z(T-1,T) = \min\{(1-\delta)U(x_{T-1}) + \delta z(T,T), z(T,T)\} \\ \dots \\ z(0,T) = \min\{(1-\delta)U(x_0) + \delta z(1,T), z(1,T)\}. \end{array}\right\} \quad (1)$$

Define the mapping $\overline{W} : \mathbf{X}_\delta \to \mathbb{R}$ by

$$\overline{W}({}_0\mathbf{x}) := \lim_{T\to\infty} z(0,T). \quad (\overline{\mathrm{W}})$$

Then, it can be shown that $\overline{W}$ is well-defined by $(\overline{\mathrm{W}})$ and satisfies (W.1)–(W.4). The details of this demonstration are presented in Appendix A.1.

Part (ii) can be demonstrated by using (W.1)–(W.4), (1) and $(\overline{\mathrm{W}})$, by studying the asymptotic behavior of any SDU SWF. This is shown in detail in Appendix A.2.

Part (iii) can be established by comparing the asymptotic behaviors of any SDU SWF with a DU SWF. This is demonstrated in detail in Appendix A.3. ■

Remark 1. Note that the particular algorithmic construction described in (1) has significance beyond the existence result described in part (i) of Theorem 1, in view of the result described in part (iii). That is, $\overline{W}$ yields an *upper bound* on SDU welfare for all consumption streams. Further, it is worth observing that there does exist a class of consumption streams $\mathbf{X}_\delta$, and an SDU SWF W on $\mathbf{X}_\delta$ such that $W({}_0x) < \overline{W}({}_0x)$ for some ${}_0x \in \mathbf{X}_\delta$. A concrete example demonstrating this non-uniqueness result is presented in Appendix A.3. In contrast, when the class of consumption streams is bounded, this possibility is ruled out, as noted in the uniqueness result of part (ii) of Theorem 1.

Any SDU SWF is a *sustainable recursive* SWF, as defined by Asheim et al. (2009), who provide an axiomatization of this class of SWFs. We record this observation (which is verified in Appendix A.4) formally as follows.

Proposition 1. *Let $\delta \in (0,1)$ and $U : \mathbb{R}_+ \to \mathbb{R}$ satisfying (U.1) be given. Then any SWF, $W : \mathbf{X}_\delta \to \mathbb{R}$, satisfying (W.1)–(W.4), is a sustainable recursive SWF.*

The following result provides a basic relationship between SDU and DU SWFs.

Proposition 2. *Assume that W is an SDU SWF.*

(i) *If ${}_0\mathbf{x} \in \mathbf{X}_\delta$, then, for all $t \geq 0$, $W({}_0\mathbf{x}) \leq W({}_t\mathbf{x}) \leq w({}_t\mathbf{x})$*
(ii) *If ${}_0\mathbf{x} \in \mathbf{X}_\delta$ is a non-decreasing stream, then $W({}_0\mathbf{x}) = w({}_0\mathbf{x})$.*

Proof. Part (i). It follows from (W.1) that for all $t \geq 0$,

$$W({}_t\mathbf{x}) = \min\{(1-\delta)U(x_t) + \delta W({}_{t+1}\mathbf{x}), W({}_{t+1}\mathbf{x})\} \leq W({}_{t+1}\mathbf{x}).$$

Hence, $W({}_0\mathbf{x}) \leq W({}_t\mathbf{x})$.

Using (W.1), we have for all $t \geq 0$,

$$W({}_t\mathbf{x}) = \min\{(1-\delta)U(x_t) + \delta W({}_{t+1}\mathbf{x}), W({}_{t+1}\mathbf{x})\} \leq (1-\delta)U(x_t) + \delta W({}_{t+1}\mathbf{x}).$$

Thus, by using (W.4), we obtain for all $t \geq 0$,

$$W({}_t\mathbf{x}) \leq (1-\delta)\sum_{s=t}^{\infty}\delta^{s-t}U(x_s) \equiv w({}_t\mathbf{x}).$$

Part (ii). Given any $t \geq 0$, we have:

$$\begin{aligned} W({}_{t+1}\mathbf{x}) &= W(x_{t+1}, x_{t+2}, \dots) \\ &\geq W(x_t, x_t, \dots) \quad \text{by (W.3) since } {}_t\mathbf{x} \text{ is non-decreasing} \\ &= U(x_t) \quad \text{by (W.2).} \end{aligned}$$

Using this in (W.1), we have for all $t \geq 0$,

$$W({}_t\mathbf{x}) = (1-\delta)U(x_t) + \delta W({}_{t+1}\mathbf{x})$$

so that:

$$W({}_0\mathbf{x}) = (1-\delta)\sum_{t=0}^{\infty}\delta^t U(x_t)$$

by using (W.4). Thus, $W({}_0\mathbf{x}) = w({}_0\mathbf{x})$. ■

3. Sustainable discounted utilitarian optimum

We now introduce the notions of feasibility and optimum in our study. Let $\mathbb{X} \subset \mathbf{X}_\delta$ denote the set of *feasible consumption streams*; it will be assumed to be non-empty and convex. This set will be determined by the technology available over time to transform inputs into outputs, and on the initial stocks of the various inputs available to an economy. In the next two sections, we will see how the set of feasible consumption streams is obtained, starting with the more primitive information of technology and available resources.

Given a discount factor δ and utility function U satisfying (U.1) and (U.2), a consumption stream ${}_0\bar{\mathbf{x}} \in \mathbb{X}$ will be called *SDU optimum* if, for some $W : \mathbf{X}_\delta \to \mathbb{R}$ satisfying (W.1)–(W.4):

$$W({}_0\mathbf{x}) \leq W({}_0\bar{\mathbf{x}}) \quad \text{for all } {}_0\mathbf{x} \in \mathbb{X}.$$

This definition entails that ${}_0\bar{\mathbf{x}} \in \mathbb{X}$ is a *unique* SDU optimum if and only if, for *every* $W : \mathbf{X}_\delta \to \mathbb{R}$ satisfying (W.1)–(W.4):

$$W({}_0\mathbf{x}) < W({}_0\bar{\mathbf{x}}) \quad \text{for all } {}_0\mathbf{x} \in \mathbb{X},\ {}_0\mathbf{x} \neq {}_0\bar{\mathbf{x}}.$$

Similarly, a consumption stream ${}_0\mathbf{x}' \in \mathbb{X}$ will be called *DU optimum* if:

$$w({}_0\mathbf{x}) \leq w({}_0\mathbf{x}') \quad \text{for all } {}_0\mathbf{x} \in \mathbb{X}.$$

We now provide a convenient sufficient condition for an egalitarian consumption stream to be the unique SDU optimum.

Proposition 3. *Let ${}_0\mathbf{x}^e \gg 0$ be an egalitarian consumption stream in $\mathbb{X}$. Assume that there exists a price sequence ${}_0\mathbf{p} = (p_0, p_1, p_2, \ldots) \gg 0$ satisfying*

$$p_{t+1}/p_t \geq \delta \quad \textit{for } t \geq 0, \tag{2}$$

$$\infty > \sum_{t=0}^{\infty} p_t x_t^e \geq \sum_{t=0}^{\infty} p_t x_t \tag{3}$$

for every consumption stream ${}_0\mathbf{x} \in \mathbb{X}$. Then ${}_0\mathbf{x}^e$ is the unique SDU optimum.

Proof. Suppose that ${}_0\mathbf{x}$ is a feasible consumption stream, distinct from ${}_0\mathbf{x}^e$, with $W({}_0\mathbf{x}) \geq W({}_0\mathbf{x}^e)$ for some $W : \mathbf{X}_\delta \to \mathbb{R}$ satisfying (W.1)–(W.4). Then, by (W.3) and Proposition 2,

$$w({}_t\mathbf{x}^e) = w({}_0\mathbf{x}^e) = U(x_0^e) = W({}_0\mathbf{x}^e) \leq W({}_0\mathbf{x}) \leq W({}_t\mathbf{x}) \leq w({}_t\mathbf{x}). \tag{4}$$

For $t \geq 0$, write

$$A_t := \sum_{\tau=t}^{\infty} \delta^\tau (x_\tau - x_\tau^e), \tag{5}$$

where the infinite sum in (5) is absolutely convergent and therefore convergent, given that ${}_0\mathbf{x} \in \mathbb{X} \subseteq \mathbf{X}_\delta$. Thus, $A_t \in \mathbb{R}$ for $t \geq 0$.

Using (U.1)–(U.2) and the fact that ${}_0\mathbf{x}^e \gg 0$ is egalitarian, we have for $\tau \geq 0$,

$$U(x_\tau) - U(x_\tau^e) \leq U'(x_\tau^e)(x_\tau - x_\tau^e) = U'(x_0^e)(x_\tau - x_\tau^e), \tag{6}$$

with strict inequality in (6) if $x_\tau \neq x^e_\tau$. Also, for $t \geq 0$,

$$w({}_t\mathbf{x}) - w({}_t\mathbf{x}^e) = \frac{1-\delta}{\delta^t} \cdot \sum_{\tau=t}^{\infty} \delta^\tau \big(U(x_\tau) - U(x^e_\tau)\big). \tag{7}$$

Combining (5)–(7), we have

$$w({}_t\mathbf{x}) - w({}_t\mathbf{x}^e) \leq \frac{1-\delta}{\delta^t} \cdot U'(x^e_0) A_t \tag{8}$$

for $t \geq 0$, with strict inequality in (8) for $t = 0$. Combining (4) and (5), we have

$$A_0 > 0 \quad \text{and} \quad A_t \geq 0 \quad \text{for all } t \geq 1. \tag{9}$$

For $t \geq 0$, write

$$a_t := \delta^t (x_t - x^e_t), \qquad b_t := \frac{p_t}{\delta^t}. \tag{10}$$

Note that, by (4) and (10), $A_t - A_{t+1} = a_t$ for all $t \geq 0$, and, by (2),

$$b_{t+1} - b_t = \frac{p_{t+1}}{\delta^{t+1}} - \frac{p_t}{\delta^t} = \frac{p_t}{\delta^{t+1}} \cdot \left(\frac{p_{t+1}}{p_t} - \delta\right) \geq 0 \tag{11}$$

for all $t \geq 0$. Then, for all $T \geq 0$, we have (using Abel's partial summation method)

$$\begin{aligned} \sum_{t=0}^{\mathrm{T}} a_t b_t &= (A_0 - A_1) b_0 + \cdots + (A_T - A_{T+1}) b_T \\ &= A_0 b_0 + A_1 (b_1 - b_0) + \cdots + A_T (b_T - b_{T-1}) - A_{T+1} b_T \\ &\geq A_0 b_0 - A_{T+1} b_T, \end{aligned} \tag{12}$$

where the inequality in (12) follows from (9) and (11). For $T \geq 0$, we get

$$\begin{aligned} A_{T+1} b_T &= \left(\sum_{\tau=T+1}^{\infty} \delta^\tau (x_\tau - x^e_\tau)\right) \cdot \frac{p_T}{\delta^{\mathrm{T}}} \\ &= \delta p_T \cdot \left[\left(\sum_{\tau=T+1}^{\infty} \delta^{\tau-(T+1)} x_\tau\right) - \frac{x^e_{T+1}}{1-\delta}\right] < \sum_{\tau=T+1}^{\infty} p_\tau x_\tau \end{aligned} \tag{13}$$

since $x^e_\tau = x^e_{T+1} > 0$ and $p_\tau / p_T \geq \delta^{\tau - T}$ for all $\tau > T$. By (3), $\lim_{T\to\infty} \sum_{\tau=T+1}^{\infty} p_\tau x_\tau = 0$. Using this fact in (13), we obtain

$$\lim_{T\to\infty} A_{T+1} b_T = 0. \tag{14}$$

It follows from (9) and (14) that, for any $\varepsilon \in (0, A_0 b_0)$, there exists $\tilde{T}$ such that, for all $T \geq \tilde{T}$, $A_{T+1} b_T \leq A_0 b_0 - \varepsilon$. Hence, by (10) and (12), for all $T \geq \tilde{T}$,

$$\sum_{t=0}^{\mathrm{T}} p_t (x_t - x^e_t) = \sum_{t=0}^{\mathrm{T}} a_t b_t \geq A_0 b_0 - A_{T+1} b_T \geq \varepsilon > 0.$$

This contradicts (3) and shows that there is no feasible stream ${}_0\mathbf{x}$, distinct from ${}_0\mathbf{x}^e$, with $W({}_0\mathbf{x}) \geq W({}_0\mathbf{x}^e)$. ■

4. Ramsey technologies

A Ramsey technology (following Ramsey, 1928) is determined by a sequence of production functions ${}_0\mathbf{g} = (g_0, g_1, g_2, \ldots)$ where, for each t, $g_t : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies

$$g_t \text{ is concave, continuous and increasing on } \mathbb{R}_+, \tag{g.1}$$

$$g_t \text{ is continuously differentiable on } \mathbb{R}_{++}, \tag{g.2}$$

$$g_t(0) = 0, \quad g_t' > 0 \quad \text{on } \mathbb{R}_{++}. \tag{g.3}$$

For each t, the *gross output function* f_t is defined by $f_t(k) = g_t(k) + k$ for all $k \geq 0$.

Let y denote *gross output*, which is split into *consumption* x and *capital input* k. A program $({}_t\mathbf{y}, {}_t\mathbf{k})$ is y_t*-feasible* if there exist ${}_t\mathbf{k}$ and ${}_{t+1}\mathbf{y}$ satisfying

$$0 \leq k_\tau \leq y_\tau \quad \text{and} \quad 0 \leq y_{\tau+1} \leq f_\tau(k_\tau) \quad \text{for all } \tau \geq t.$$

The consumption ${}_t\mathbf{x}$ associated with a y_t-feasible program $({}_t\mathbf{y}, {}_t\mathbf{k})$ is defined by $x_\tau = y_\tau - k_\tau$ for all $\tau \geq t$. A y_t-feasible program $({}_t\mathbf{y}, {}_t\mathbf{k})$ is called *egalitarian* if the consumption stream ${}_t\mathbf{x}$ associated with it is egalitarian. A y_t-feasible program $({}_t\bar{\mathbf{y}}, {}_t\bar{\mathbf{k}})$ is y_t*-efficient* if there is no y_t-feasible program $({}_t\mathbf{y}, {}_t\mathbf{k})$ satisfying $x_\tau \geq \bar{x}_\tau$ for all $\tau \geq t$, with strict inequality for some $\tau \geq t$.

The set $\mathbb{X} \subset \mathbb{R}_+^{\mathbb{Z}_+}$ of feasible consumption streams, introduced in the previous section, can be described for Ramsey technologies by:

$$\mathbb{X} = \{{}_0\mathbf{x} \in \mathbb{R}_+^{\mathbb{Z}_+} \mid {}_0\mathbf{x} \text{ is a consumption stream associated with a } y_0\text{-feasible program } ({}_0\mathbf{y}, {}_0\mathbf{k})\}.$$

Combined with the results of Cass and Yaari (1971), Proposition 3 implies the following sufficient condition for a unique SDU optimum.

Proposition 4. *Consider a Ramsey technology satisfying (g.1)–(g.3) and $\mathbb{X} \subseteq \mathbf{X}_\delta$. Assume that the y_0-feasible program $({}_0\mathbf{y}^e, {}_0\mathbf{k}^e)$ is egalitarian and y_0-efficient with $({}_0\mathbf{y}^e, {}_0\mathbf{k}^e) \gg 0$, and satisfies:*

$$\delta f_t'(k_t^e) \leq 1 \quad \textit{for } t \geq 0, \tag{15}$$

$$\lim_{T\to\infty} \sum_{t=0}^{T} \left[\frac{1}{\prod_{\tau=0}^{t} f_\tau'(k_\tau^e)} \right] < \infty. \tag{16}$$

Then $({}_0\mathbf{y}^e, {}_0\mathbf{k}^e)$ is the unique SDU optimum.

Proof. Since ${}_0\mathbf{k}^e \gg 0$, the price sequence ${}_0\mathbf{p} \gg 0$ determined by

$$p_0 = 1 \quad \text{and} \quad p_{t+1} f_t'(k_t^e) = p_t \quad \text{for all } t \geq 0 \tag{17}$$

is well-defined. Then (15) implies that (2) is satisfied and (16) implies that $\sum_{t=0}^{\infty} p_t x_t^e < \infty$, and so (3) follows from the Corollary of Cass and Yaari (1971, p. 338). Hence, Proposition 4 follows from Proposition 3. ■

We illustrate the usefulness of Proposition 4 by considering the special case of a linear technology where, for each t, $g_t(k) = r_t k$ with $r_t > 0$, so that $f_t(k) = (r_t + 1)k$. In this case, the price sequence ${}_0\mathbf{p}$ defined by (17) is independent of the program, as for any $({}_0\mathbf{y}, {}_0\mathbf{k})$, $p_{t+1}(r_t + 1) = p_t$ for all $t \geq 0$. Furthermore, the set of y_t-feasible consumption streams at time t is given by $\{{}_t\mathbf{x} \mid \sum_{\tau=t}^{\infty} p_\tau x_\tau \leq p_t y_t\}$. Assume that

$$\sum_{t=0}^{\infty} p_t = \lim_{T\to\infty} \sum_{t=0}^{T} \left[\frac{1}{\prod_{\tau=0}^{t} (r_\tau + 1)} \right] < \infty; \tag{18}$$

$r_t \geq \epsilon > 0$ for all $t \geq 0$ is sufficient for this. Moreover, assume also that

$$\delta p_0/p_1 < 1, \tag{p.1}$$

$$\delta p_1/\left((1-\delta)\sum_{t=2}^{\infty} p_t\right) > 1, \tag{p.2}$$

$$\delta p_t/p_{t+1} \leq 1 \quad \text{for } t \geq 2, \tag{p.3}$$

implying that the economy is particularly productive in period 1.

By combining (p.3) and (18) with Proposition 4, it follows that, for given $y_2 > 0$, the unique SDU optimum consumption stream at $t = 2$, ${}_2\mathbf{x}^*$, satisfies $x_t^* = \bar{x}_2(y_2) = p_2 y_2/(\sum_{\tau=2}^{\infty} p_\tau)$ for all $t \geq 2$. By (U.1), the value function $V_2(y_2) \equiv U(\bar{x}_2(y_2)) = W({}_2\mathbf{x}^*)$ is strictly concave and continuously differentiable with

$$V_2'(y_2) = \left(p_2/\sum_{t=2}^{\infty} p_t\right) U'(\bar{x}_2(y_2)) > 0.$$

To determine, for given $y_1 > 0$, the unique SDU optimum consumption stream at $t = 1$, ${}_1\mathbf{x}^*$, note first that $y_2 = p_1(y_1 - x_1)/p_2$, and that there is a unique consumption at time 1, $x_1 = \bar{x}_1(y_1) = p_1 y_1/(\sum_{\tau=1}^{\infty} p_\tau) > 0$, satisfying $U(x_1) = V_2(p_1(y_1 - x_1)/p_2)$.

For given $y_1 > 0$, consider the problem of maximizing w.r.t. x_1

$$\pi_1(x_1) \equiv (1-\delta)U(x_1) + \delta V_2(p_1(y_1 - x_1)/p_2) \tag{19}$$

subject to $x_1 \leq \bar{x}_1(y_1)$. By (U.1)–(U.2) and (p.2), $\pi_1'(x_1)$ is continuous and decreasing with $\pi_1'(x_1) > 0$ for all x_1 close to 0 and $\pi_1'(x_1) < 0$ for $x_1 = \bar{x}_1(y_1)$. So there is a unique $x_1^* \in (0, \bar{x}_1(y_1))$ such that $\pi_1'(x_1^*) = 0$. Since π_1 is strictly concave, x_1^* uniquely solves maximization problem (19).

It follows from (W.1) that $x_1(y_1) = x_1^* \in (0, \bar{x}_1(y_1))$ is the SDU optimum consumption at $t = 1$ as a function of y_1, with $x_1(y_1)$ satisfying the first-order condition

$$(1-\delta)U'(x_1(y_1)) = \delta \frac{p_1}{\sum_{\tau=2}^{\infty} p_\tau} U'(\bar{x}_2(p_1(y_1 - x_1(y_1)))). \tag{20}$$

By (U.1), the value function $V_1(y_1) \equiv \pi_1(x_1(y_1))$ is strictly concave and, by Benveniste and Scheinkman (1979, Lemma 1), continuously differentiable with

$$V_1'(y_1) = (1-\delta)U'(x_1(y_1)) > 0.$$

Note that $x_1^* = x_1(y_1) < \bar{x}_1(y_1) < \bar{x}_2(p_1(y_1 - x_1(y_1))/p_2) = x_t^*$ for all $t \geq 2$.

To determine, for given $y_0 > 0$, the unique SDU optimum consumption stream at $t = 0$, ${}_0\mathbf{x}^*$, note first that $y_1 = p_0(y_0 - x_0)/p_1$, and that there is a unique consumption at time 0, $x_0 = \bar{x}_0(y_0) > 0$, satisfying $U(x_0) = V_1(p_0(y_0 - x_0)/p_1)$.

For given $y_0 > 0$, consider the problem of maximizing w.r.t. x_0

$$\pi_0(x_0) \equiv (1-\delta)U(x_0) + \delta V_1(p_0(y_0 - x_0)/p_1) \tag{21}$$

subject to $x_0 \leq \bar{x}_0(y_0)$. By (U.1)–(U.2), $\pi_0'(x_0)$ is continuous and decreasing with $\pi_0'(x_1) > 0$ for all x_0 close to 0. So there is a unique $x_0^* \in (0, \bar{x}_0(y_0)]$ such that

$$\pi_0'(x_0^*) \geq 0 \quad \text{and} \quad \pi_0'(x_0^*)[x_0^* - \bar{x}_0(y_0)] = 0.$$

Since π_0 is strictly concave, x_0^* uniquely solves maximization problem (21).

It follows from (W.1) that $x_0(y_0) = x_0^* \in (0, \bar{x}_0(y_0)]$ is the SDU optimum consumption at $t = 0$ as a function of y_0. Either (i) $x_0(y_0) = \bar{x}_0(y_0)$ or (ii) $x_0(y_0) < \bar{x}_0(y_0)$, in which case $x_0(y_0)$ satisfies the first-order condition

$$U'(x_0(y_0)) = \delta \frac{p_0}{p_1} U'(x_1(p_0(y_0 - x_0)/p_1)). \tag{22}$$

The definition of $\bar{x}_0(y_0)$ implies that $x_1^* < x_0^* < x_t^*$ for all $t \geq 2$ in case (i), while this follows from (p.1) and the strict concavity of U in case (ii).

This example gives rise to two remarks. First, even though it follows from Proposition 2 that SDU welfare is non-decreasing: $W({}_t\mathbf{x}) \leq W({}_{t+1}\mathbf{x})$ for all $t \geq 0$, it is not the case that an SDU optimum consumption stream must be non-decreasing. Indeed, x_0 may contribute to $W({}_0\mathbf{x})$ even if $x_0 > x_1$, provided that $U(x_1) < W({}_2\mathbf{x})$. The above example with a non-stationary technology illustrates this possibility.

Second, the example illustrates that SDU does not satisfy Finite Anonymity. Since $r_0 > 0$, so that $1 = p_0 > p_1$, it is feasible to permute the consumption levels of generations 0 and 1. However, this leads to strictly lower SDU welfare. Due to utility discounting, generation 1 makes a larger sacrifice than generation 0 for the purpose of accumulating capital to benefit later generations (cf. (22)). On the other hand, utility discounting also reduces the sacrifice that should optimally be made (cf. (20)), thereby protecting generation 1 from an excessively high savings rate.

We now specialize our discussion to the case in which the production functions for the various time periods are the same, and the net capital productivity approaches zero as the capital stock approaches infinity. This is expressed formally in

$$g_t = g \quad \text{for all } t \geq 0, \tag{g.4}$$

$$\lim_{k\to\infty} g'(k) = 0. \tag{g.5}$$

Write the gross output function as $f(k) = g(k) + k$.

It follows from (g.1)–(g.5) that, for every $y > 0$, there exists a unique $x(y)$, satisfying $0 < x(y) < y$, which solves $y = f(y - x(y))$; define $x(0) = 0$. For each y, $x(y)$ represents the consumption level which keeps the output level y intact over time. Clearly, $x : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous for $x \geq 0$, and differentiable with

$$x'(y) = \frac{f'(y - x(y)) - 1}{f'(y - x(y))} > 0.$$

For all $y > 0$, write

$$\delta(y) := \frac{1}{f'(y - x(y))}.$$

Then $\delta : \mathbb{R}_{++} \to (0, 1)$ is continuous and non-decreasing in y with $\lim_{y\to\infty} \delta(y) = 1$ by (g.5). Define $\delta(0) := \lim_{y\downarrow 0} \delta(y)$.

Finally, we can define $y^\infty(\delta)$, for all $\delta \in (0, 1)$, by

$$y^\infty(\delta) := \min\{y \geq 0 \mid \delta(y) \geq \delta\}.$$

Then $y^\infty : (0, 1) \to \mathbb{R}_+$ is strictly increasing on $[\delta(0), 1]$.

Theorem 2. *Consider a Ramsey technology satisfying (g.1)–(g.5). For any $\delta \in (0, 1)$ and $y_0 > 0$, there exists a unique SDU optimum ${}_0\mathbf{x}^*$.*

(i) *If $y_0 \geq y_\infty(\delta)$, then ${}_0\mathbf{x}^*$ is efficient and egalitarian with $x_t^* = x(y_0)$ for all $t \geq 0$.*
(ii) *If $y_0 < y_\infty(\delta)$, then ${}_0\mathbf{x}^*$ is efficient and strictly increasing, maximizing $w({}_0\mathbf{x})$ over all y_0-feasible consumption streams and converging to $x(y_\infty(\delta))$.*

For the proof of Theorem 2 we use the result that, for any $\delta \in (0, 1)$, the set of y_0-feasible consumption streams, $\mathbb{X}$, is included in $\mathbf{X}_\delta$. This result is stated in Lemma 1 below, and proved in Appendix A.5.

Lemma 1. *Let $y_0 > 0$ be given. For all $\delta \in (0, 1)$, $\mathbb{X} \subseteq \mathbf{X}_\delta$.*

Proof of Theorem 2. Fix $\delta \in (0, 1)$ and $y_0 > 0$.

Case (i): $y_0 \geq y_\infty(\delta)$. By the definition of $y_\infty(\delta)$ it follows that $\delta(y_0) \geq \delta$. Consider the y_0-feasible stream ${}_0\mathbf{x}^*$ defined by $x_t^* = x(y_0)$ for all $t \geq 0$, with associated y_0-feasible program $({}_0\mathbf{y}^e, {}_0\mathbf{k}^e)$ satisfying, for all $t \geq 0$, $y_t^e = y_0$ and $k_t^e = y_0 - x(y_0)$. Then, $({}_0\mathbf{y}^e, {}_0\mathbf{k}^e)$ is clearly egalitarian.

Since $y_0 > 0$, we have $f(y_0 - x(y_0)) = y_0 > 0$, and so $(y_0 - x(y_0)) > 0$. Thus, $\theta := g'(y_0 - x(y_0))$ is well-defined and positive. Hence,

$$f'(k_t^e) = f'(y_0 - x(y_0)) = 1 + \theta > 1$$

for all t, so that (16) is satisfied. Further, the price sequence ${}_0\mathbf{p} \gg 0$ determined by (17), is well-defined, and $\lim_{t\to\infty} p_t k_t^e = 0$. Thus, by the Theorem of Cass and Yaari (1971, p. 337), $({}_0\mathbf{y}^e, {}_0\mathbf{k}^e)$ is efficient. By the definition of the function δ,

$$f'(k_t^e) = f'(y_0 - x(y_0)) = \frac{1}{\delta(y_0)} \le \frac{1}{\delta}$$

for all t, so that (15) is also satisfied. It follows now from Proposition 4 and Lemma 1 that ${}_0\mathbf{x}^*$ is the unique SDU optimum.

Case (ii): $y_0 < y_\infty(\delta)$. By the definition of $y_\infty(\delta)$ it follows that $\delta(y_0) < \delta$. It is well-known (see Beals and Koopmans, 1969) that there exists a y_0-feasible program $({}_0\mathbf{y}^*, {}_0\mathbf{k}^*)$ satisfying

$$\lim_{t\to\infty} y_t^* = y_\infty(\delta) \quad \text{and} \quad \lim_{t\to\infty} k_t^* = y_\infty(\delta) - x(y_\infty(\delta)),$$

which is efficient, and which has associated with it a y_0-feasible stream ${}_0\mathbf{x}^* \in \mathbf{X}_\varphi$. Furthermore, ${}_0\mathbf{x}^*$ is strictly increasing and uniquely maximizes $w({}_0\mathbf{x})$ over all y_0-feasible programs $({}_0\mathbf{y}, {}_0\mathbf{k})$ with associated y_0-feasible stream ${}_0\mathbf{x}$. Hence, if ${}_0\mathbf{x}$ is a y_0-feasible stream distinct from ${}_0\mathbf{x}^*$ and $W : \mathbf{X}_\delta \to \mathbb{R}$ satisfies (W.1)–(W.4), then Proposition 2 and Lemma 1 imply

$$W({}_0\mathbf{x}^*) = w({}_0\mathbf{x}^*) > w({}_0\mathbf{x}) \ge W({}_0\mathbf{x}),$$

thereby establishing that ${}_0\mathbf{x}^*$ is the unique SDU optimum. ■

Theorem 2 means that the unique SDU optimum stream coincides with the DU optimum stream with increasing consumption if there is a small initial capital stock (so that net capital productivity is high), while it coincides with the egalitarian and efficient stream with a large initial capital stock.

5. Dasgupta–Heal–Solow technologies

A Dasgupta–Heal–Solow technology (DHS) (see Dasgupta and Heal, 1974, 1979; Solow, 1974) is determined by a stationary production function $G : \mathbb{R}^3_+ \to \mathbb{R}$ that satisfies

$$G \text{ is concave, non-decreasing, homogeneous of degree one, and continuous for } (k, r, \ell) \in \mathbb{R}^3_+, \tag{G.1}$$

$$G \text{ is twice continuously differentiable and satisfies } (G_k, G_r, G_\ell) \gg 0 \text{ for } (k, r, \ell) \in \mathbb{R}^3_{++}. \tag{G.2}$$

$$G(k, 0, \ell) = 0 = G(0, r, \ell) \tag{G.3}$$

$$\text{Given any } (k', r') \gg 0, \text{ there is } \eta' > 0 \text{ such that for all } (k, r) \text{ satisfying } k \ge k', \quad 0 < r \le r', \qquad [rG_r(k, r, 1)]/G_\ell(k, r, 1) \ge \eta'. \tag{G.4}$$

(G.3) states that both capital input k and *resource use* r are essential in production. (G.4) requires that the ratio of the share of the resource in net output to the share of labor in net output is bounded away from zero (when labor is fixed at unit level).

The labor force is assumed to be stationary and normalized to 1. The *gross output function* F, is defined by $F(k, r) = G(k, r, 1) + k$ for all $(k, r) \ge 0$, and is assumed to satisfy

$$F \text{ is strictly concave in } (k, r) \text{ on } \mathbb{R}^2_+ \tag{F.1}$$

$$F_{kr} \ge 0 \quad \text{for } (k, r) \in \mathbb{R}^2_{++}, \tag{F.2}$$

where (F.2) is used to ensure (24) of Lemma 3 below.

Let y denote gross output and m the total resource stock. The production possibilities are described by the stationary transformation set $\mathcal{T}$ given by

$$\mathcal{T} = \{[(k, m), (y, m')] \mid 0 \leq y \leq F(k, r); 0 \leq r = m - m' \leq m\}.$$

A program $({}_t\mathbf{y}, {}_t\mathbf{m}, {}_t\mathbf{k})$ is (y_t, m_t)-*feasible* if there exist ${}_t\mathbf{k}$, ${}_{t+1}\mathbf{y}$ and ${}_{t+1}\mathbf{m}$ satisfying

$$0 \leq k_\tau \leq y_\tau \quad \text{and} \quad [(k_\tau, m_\tau), (y_{\tau+1}, m_{\tau+1})] \in \mathcal{T} \quad \text{for all } \tau \geq t,$$

The consumption ${}_t\mathbf{x}$ associated with a (y_t, m_t)-feasible program $({}_t\mathbf{y}, {}_t\mathbf{m}, {}_t\mathbf{k})$ is defined by $x_\tau = y_\tau - k_\tau$ for all $\tau \geq t$. A (y_t, m_t)-feasible program $({}_t\mathbf{y}, {}_t\mathbf{m}, {}_t\mathbf{k})$ is called *egalitarian* if the consumption stream ${}_t\mathbf{x}$ associated with it is egalitarian. A (y_t, m_t)-feasible program $({}_t\bar{\mathbf{y}}, {}_t\bar{\mathbf{m}}, {}_t\bar{\mathbf{k}})$ is (y_t, m_t)-*efficient* if there is no (y_t, m_t)-feasible program $({}_t\mathbf{y}, {}_t\mathbf{m}, {}_t\mathbf{k})$ satisfying $x_\tau \geq \bar{x}_\tau$ for all $\tau \geq t$, with strict inequality for some $\tau \geq t$.

The set $\mathbb{X} \subset \mathbb{R}_+^{\mathbb{Z}_+}$ of feasible consumption streams, introduced in Section 3, can be described for DHS technologies by:

$$\mathbb{X} = \{{}_0\mathbf{x} \in \mathbb{R}_+^{\mathbb{Z}_+} \mid {}_0\mathbf{x} \text{ is a consumption stream associated with a } (y_0, m_0)\text{-feasible program } ({}_0\mathbf{y}, {}_0\mathbf{m}, {}_0\mathbf{k})\}.$$

Lemma 2 below has the role in the analysis of this section as Lemma 1 had in the analysis of Section 4. Its proof is similar to that of Lemma 1 as indicated in Appendix A.5.

Lemma 2. *Let* $(y_0, m_0) \gg 0$ *be given. For all* $\delta \in (0, 1)$, $\mathbb{X} \subseteq \mathbf{X}_\delta$.

Assumptions (G.1)–(G.4) and (F.1)–(F.2) do not ensure the existence of an egalitarian stream with positive consumption. We concentrate on those technologies satisfying (G.1)–(G.4) and (F.1)–(F.2) which do. That is, we assume:

There exists from any $(y, m) \gg 0$ an egalitarian positive consumption stream. (E)

Cass and Mitra (1991) give a necessary and sufficient condition on F for (E) to hold.

Lemma 3. *Consider a DHS technology satisfying (G.1)–(G.4), (F.1)–(F.2) and (E). For any* $(y_0, m_0) \gg 0$, *there exists a unique* (y_0, m_0)*-feasible program* $({}_0\mathbf{y}^e, {}_0\mathbf{m}^e, {}_0\mathbf{k}^e)$ *such that the associated* (y_0, m_0)*-feasible stream* ${}_0\mathbf{x}^e \gg 0$ *is efficient and egalitarian. Furthermore, the price sequence* ${}_0\mathbf{p} \gg 0$ *determined by*

$$p_0 = 1 \quad \textit{and} \quad p_{t+1}F_k(k_t^e, m_t^e - m_{t+1}^e) = p_t \quad \textit{for all } t \geq 0 \tag{23}$$

satisfies:

$$0 < \frac{p_t}{p_{t-1}} < \frac{p_{t+1}}{p_t} \quad \textit{for all } t > 1 \tag{24}$$

and:

$$\infty > \sum_{t=0}^{\infty} p_t x_t^e \geq \sum_{t=0}^{\infty} p_t x_t \tag{25}$$

holds for every (y_0, m_0)*-feasible stream* ${}_0\mathbf{x}$.

For each $(y_0, m_0) \gg 0$, consider the unique (y_0, m_0)-feasible program $({}_0\mathbf{y}^e, {}_0\mathbf{m}^e, {}_0\mathbf{k}^e)$, guaranteed by Lemma 3, such that the associated (y_0, m_0)-feasible consumption stream ${}_0\mathbf{x}^e \gg 0$ is efficient and egalitarian. Furthermore, let ${}_0\mathbf{p} \gg 0$ be the associated price sequence determined by (23). By (25), we have $\sum_{t=0}^{\infty} p_t < \infty$. For each $(y_0, m_0) \gg 0$, we can then define:

$$\delta^0(y_0, m_0) := \left(\frac{p_1}{p_0}\right) \quad \text{and} \quad \delta^\infty(y_0, m_0) := \left[\frac{\sum_{t=1}^{\infty} p_t}{\sum_{t=0}^{\infty} p_t}\right].$$

For each $(y_0, m_0) \gg 0$, we refer to $\delta^0(y_0, m_0)$ as the *short-run discount factor* and to $\delta^\infty(y_0, m_0)$ as the *long-run discount factor* at time 0 supporting the efficient and egalitarian (y_0, m_0)-feasible program $({}_0\mathbf{y}^e, {}_0\mathbf{m}^e, {}_0\mathbf{k}^e)$.

When the short-run discount factor is at least as large as δ, the efficient egalitarian program described in Lemma 3 is the unique SDU optimum, as the following proposition shows.

Proposition 5. *Consider a DHS technology satisfying (G.1)–(G.4), (F.1)–(F.2) and (E). If $(y_0, m_0) \gg 0$ satisfies $\delta^0(y_0, m_0) \geq \delta$, then the efficient and egalitarian (y_0, m_0)-feasible stream ${}_0\mathbf{x}^e \gg 0$ is the unique SDU optimum.*

Proof. It follows from Lemma 3 that ${}_0\mathbf{p} \gg 0$, the price sequence determined by (23) and supporting the unique (y_0, m_0)-feasible program $({}_0\mathbf{y}^e, {}_0\mathbf{m}^e, {}_0\mathbf{k}^e)$, satisfies (2) and (3). Hence, by Proposition 3, ${}_0\mathbf{x}^e$ is the unique SDU optimum. ■

When the short-run discount factor is smaller than δ, the description of an SDU optimum is more involved. To carry out the analysis, we have to compare the *long-run discount factor* with δ. For this purpose, a preliminary result comparing the short-run and the long-run discount factors is useful.

Lemma 4. *Consider a DHS technology satisfying (G.1)–(G.4), (F.1)–(F.2) and (E). For all $(y_0, m_0) \gg 0$, $\delta^0(y_0, m_0) < \delta^\infty(y_0, m_0)$.*

To proceed further, we note that even when the short-run discount factor is initially smaller than δ for a (y_0, m_0)-feasible program, the short-run discount factor becomes at least as large as δ after a finite time period, provided the consumption stream on such a program is bounded away from zero.

Lemma 5. *Consider a DHS technology satisfying (G.1)–(G.4), (F.1)–(F.2) and (E). Let $(y_0, m_0) \gg 0$ and $\delta \in (0, 1)$. If a (y_0, m_0)-feasible program $({}_0\mathbf{y}, {}_0\mathbf{m}, {}_0\mathbf{k})$ has an associated (y_0, m_0)-feasible stream ${}_0\mathbf{x} \gg 0$ with $\liminf_{T\to\infty} w({}_T\mathbf{x}) > U(0)$, then there exists $\tau \geq 0$ such that $\delta \leq \delta^0(y_\tau, m_\tau)$.*

As shown in the example illustrating Proposition 4, streams that are *not* non-decreasing can be SDU optimum in non-stationary technologies. However, SDU optimum streams in DHS technologies (as in stationary Ramsey technologies) will in fact be streams maximizing $w({}_0\mathbf{x})$ subject to the constraint that $x_t \leq x_{t+1}$ for all $t \geq 0$. Such streams have been analyzed in discrete time by Asheim (1988) and in continuous time by Pezzey (1994). This motivates the following lemma.

Lemma 6. *Consider a DHS technology satisfying (G.1)–(G.4), (F.1)–(F.2) and (E). For any $(y_0, m_0) \gg 0$, there exists a (y_0, m_0)-feasible program $({}_0\mathbf{y}^*, {}_0\mathbf{m}^*, {}_0\mathbf{k}^*)$ with the property that the associated (y_0, m_0)-feasible stream ${}_0\mathbf{x}^* \gg 0$ maximizes $w({}_0\mathbf{x})$ over all (y_0, m_0)-feasible and non-decreasing consumption streams ${}_0\mathbf{x}$. Furthermore,*

(i) *$({}_0\mathbf{y}^*, {}_0\mathbf{m}^*, {}_0\mathbf{k}^*)$ is unique and time-consistent (for all $t \geq 0$, ${}_t\mathbf{x}^*$ maximizes $w({}_t\mathbf{x})$ over all (y_t^*, m_t^*)-feasible and non-decreasing consumption streams ${}_t\mathbf{x}$),*
(ii) *${}_0\mathbf{x}^* \in \mathbf{X}_\varphi$; in particular, there is a $\tau \geq 0$ such that $x_0^* < \cdots < x_{\tau-1}^* < x_\tau^* = x_{\tau+1}^* = \cdots$, where $\tau > 0$ if $\delta^\infty(y_0, m_0) < \delta$, and $\tau = 0$ if $\delta^\infty(y_0, m_0) \geq \delta$.*
(iii) *There is a μ such that if ${}_0\mathbf{x}$ is an arbitrary (y_0, m_0)-feasible stream, with ${}_1\mathbf{x}$ non-decreasing, then*

$$\delta \cdot [w({}_1\mathbf{x}) - w({}_1\mathbf{x}^*)] \leq \mu \cdot [U(x_0^*) - U(x_0)] \tag{26}$$

where $\mu = 1$ if $\delta^\infty(y_0, m_0) < \delta$ and $0 < \mu \leq 1$ if $\delta^\infty(y_0, m_0) \geq \delta$, and where (26) is strict if the associated (y_0, m_0)-feasible program $({}_0\mathbf{y}, {}_0\mathbf{m}, {}_0\mathbf{k})$ is distinct from $({}_0\mathbf{y}^*, {}_0\mathbf{m}^*, {}_0\mathbf{k}^*)$.

Lemma 6 entails that there exist unique policy functions k^* and m^* such that, for all $(y_0, m_0) \gg 0$, $k_0^* = k^*(y_0, m_0)$, $m_1^* = m^*(y_0, m_0)$ and $y_1^* = F(k^*(y_0, m_0), m_0 - m^*(y_0, m_0))$, where $({}_0\mathbf{y}^*, {}_0\mathbf{m}^*, {}_0\mathbf{k}^*)$ is the unique (y_0, m_0)-feasible program with the property that the associated (y_0, m_0)-feasible stream ${}_0\mathbf{x}^* \gg 0$ maximizes $w({}_0\mathbf{x})$ over all (y_0, m_0)-feasible and non-decreasing consumption streams ${}_0\mathbf{x}$.

Theorem 3. *Consider a DHS technology satisfying (G.1)–(G.4), (F.1)–(F.2) and (E). For any $\delta \in (0, 1)$ and $(y_0, m_0) \gg 0$, let ${}_0\mathbf{x}^* \gg 0$ denote the efficient (y_0, m_0)-feasible stream maximizing $w({}_0\mathbf{x})$ over all (y_0, m_0)-feasible and non-decreasing consumption streams ${}_0\mathbf{x}$. Then ${}_0\mathbf{x}^*$ is the unique SDU optimum. The stream has an eventual egalitarian phase, preceded by a phase with increasing consumption if and only if $\delta^\infty(y_0, m_0) < \delta$.*

Proof. Suppose that ${}_0\mathbf{x}$ is a (y_0, m_0)-feasible stream distinct from ${}_0\mathbf{x}^*$ such that $W({}_0\mathbf{x}) \geq W({}_0\mathbf{x}^*)$ for some $W : \mathbf{X}_\delta \to \mathbb{R}$ satisfying (W.1)–(W.4). Let $({}_0\mathbf{y}, {}_0\mathbf{m}, {}_0\mathbf{k})$ be the (y_0, m_0)-feasible program associated with ${}_0\mathbf{x}$. Since, by Theorem 1(iii) and Proposition 2(ii) (recalling that ${}_0\mathbf{x}^*$ is non-decreasing),

$$\overline{W}({}_0\mathbf{x}) \geq W({}_0\mathbf{x}) \geq W({}_0\mathbf{x}^*) = w({}_0\mathbf{x}^*) > U(0),$$

it follows from $(\overline{\mathrm{W}})$ that $\liminf_{T\to\infty} w({}_T\mathbf{x}) > U(0)$. Hence, by Lemma 5, there exists $\tilde{\tau} \geq 0$ such that $\delta^0(y_{\tilde{\tau}}, m_{\tilde{\tau}}) \geq \delta$. By Proposition 5 and (W.1), we may assume, without loss of generality, that $({}_{\tilde{\tau}}\mathbf{y}, {}_{\tilde{\tau}}\mathbf{m}, {}_{\tilde{\tau}}\mathbf{k}) = ({}_{\tilde{\tau}}\mathbf{y}^e, {}_{\tilde{\tau}}\mathbf{m}^e, {}_{\tilde{\tau}}\mathbf{k}^e)$, where $({}_{\tilde{\tau}}\mathbf{y}^e, {}_{\tilde{\tau}}\mathbf{m}^e, {}_{\tilde{\tau}}\mathbf{k}^e)$ is the unique efficient and egalitarian $(y_{\tilde{\tau}}, m_{\tilde{\tau}})$-feasible program. By Lemmas 4 and 6(i)&(ii), $k_t = k^*(y_t, m_t)$, $m_{t+1} = m^*(y_t, m_t)$ and $y_{t+1} = F(k^*(y_t, m_t), m_t - m^*(y_t, m_t))$ for all $t \geq \tilde{\tau}$. Since ${}_0\mathbf{x}$ is distinct from ${}_0\mathbf{x}^*$, we may define $\tau \geq 0$ by

$$\begin{aligned}\tau := \max\{t \geq 0 \mid\ & k_t \neq k^*(y_t, m_t) \text{or } m_{t+1} \neq m^*(y_t, m_t)\\ & \text{or } y_{t+1} \neq F(k^*(y_t, m_t), m_t - m^*(y_t, m_t))\}.\end{aligned}$$

Let $({}_\tau\mathbf{y}^*, {}_\tau\mathbf{m}^*, {}_\tau\mathbf{k}^*)$ be the unique (y_τ, m_τ)-feasible program with the property that the associated (y_τ, m_τ)-feasible stream ${}_\tau\mathbf{x}^* \gg 0$ maximizes $w({}_\tau\mathbf{x}')$ over all (y_τ, m_τ)-feasible and non-decreasing consumption streams ${}_\tau\mathbf{x}'$. By the definition of τ, $({}_\tau\mathbf{y}, {}_\tau\mathbf{m}, {}_\tau\mathbf{k})$ is distinct from $({}_\tau\mathbf{y}^*, {}_\tau\mathbf{m}^*, {}_\tau\mathbf{k}^*)$ with ${}_{\tau+1}\mathbf{x}$ being non-decreasing. By (W.1), we may assume, without loss of generality, that $W({}_\tau\mathbf{x}) \geq W({}_\tau\mathbf{x}^*) \geq 0$. By Lemma 6(iii),

$$W({}_\tau\mathbf{x}) - W({}_\tau\mathbf{x}^*) \leq w({}_\tau\mathbf{x}) - w({}_\tau\mathbf{x}^*) < (1 - \mu) \cdot [U(x_\tau) - U(x^*_\tau)], \tag{27}$$

where $\mu = 1$ if $\delta^\infty(y_\tau, m_\tau) < \delta$ and $0 < \mu \leq 1$ if $\delta^\infty(y_\tau, m_\tau) \geq \delta$, since $W({}_\tau\mathbf{x}) \leq w({}_\tau\mathbf{x})$ by Proposition 2(i) and $W({}_\tau\mathbf{x}^*) = w({}_\tau\mathbf{x}^*)$ by Proposition 2(ii), keeping in mind that ${}_\tau\mathbf{x}^* \in \mathbf{X}_\varphi$ is non-decreasing.

Case 1: $\delta^\infty(y_\tau, m_\tau) < \delta$. Then, by Lemma 6(iii), $\mu = 1$, implying by (27) that, $W({}_\tau\mathbf{x}) - W({}_\tau\mathbf{x}^*) < 0$. This contradicts $W({}_\tau\mathbf{x}) \geq W({}_\tau\mathbf{x}^*)$.

Case 2: $\delta^\infty(y_\tau, m_\tau) \geq \delta$. By Lemma 6(ii), ${}_\tau\mathbf{x}^*$ is egalitarian, implying that $W({}_\tau\mathbf{x}^*) = w({}_\tau\mathbf{x}^*) = w({}_{\tau+1}\mathbf{x}^*)$. Furthermore, it follows from Proposition 2(i) that $W({}_\tau\mathbf{x}) \leq W({}_{\tau+1}\mathbf{x}) \leq w({}_{\tau+1}\mathbf{x}^*)$. Hence, by Lemma 6(iii),

$$W({}_\tau\mathbf{x}) - W({}_\tau\mathbf{x}^*) \leq w({}_{\tau+1}\mathbf{x}) - w({}_{\tau+1}\mathbf{x}^*) < \frac{\mu \cdot [U(x^*_\tau) - U(x_\tau)]}{\delta}, \tag{28}$$

where $0 < \mu \leq 1$. If $\mu = 1$, then (27) contradicts $W({}_\tau\mathbf{x}) \geq W({}_\tau\mathbf{x}^*)$. If $0 < \mu < 1$, then (27) and (28) are incompatible.

In either case, we contradict that there exists a (y_0, m_0)-feasible stream ${}_0\mathbf{x}$ distinct from ${}_0\mathbf{x}^*$ such that $W({}_0\mathbf{x}) \geq W({}_0\mathbf{x}^*) \geq 0$.

It follows from Lemma 6(ii) that ${}_0\mathbf{x}^*$ has an eventual egalitarian phase, preceded by a phase with increasing consumption if and only if $\delta^\infty(y_0, m_0) < \delta$. ■

6. Concluding remarks

The DHS model of capital accumulation and resource depletion gives rise to interesting distributional conflicts. On the one hand, when applied to DHS technologies DU undermines the interests of the generations in the far future by forcing consumption to approach zero as time goes to infinity. On the other hand, criteria like classical utilitarianism and leximin that treat generations equally by satisfying Finite Anonymity, and thus are not numerically representable, lead to consequences that may not be compelling: classical utilitarianism leads to unbounded inequality by giving rise to unlimited growth, while leximin does not allow for any trade-off between the interests of different generations, meaning that poverty may be perpetuated if the economy has a small initial endowment of stocks (cf. Solow, 1974).

In this paper we have applied *sustainable discounted utilitarianism* (SDU) to DHS technologies and showed that the application of this criterion resolves in an appealing way the distributional conflicts that arise in this class of technologies:

(1) It allows for growth and development initially when the economy is highly productive.
(2) It leads to an efficient and egalitarian stream eventually when resource depletion and capital accumulation have reduced net capital productivity. By thus preventing consumption to approach zero, it respects the interests of future generations. By not yielding unlimited growth, it ensures bounded inequality.

We have also applied SDU to the usual one-sector model of economic growth (Ramsey technologies). If, in this setting, there is a small initial capital stock (so that net capital productivity is high), then the criterion leads to the DU optimum stream with increasing consumption. With a large initial capital stock, however, the criterion gives rise to an efficient and egalitarian stream.

SDU trades off present and future consumption if and only if the present is worse off than the future, while it gives priority to the interests of future generations otherwise. In the two classes of technologies considered, this property of SDU entails that the criterion allows for economic development when productivity is high without leading to inequitable outcomes. A dilemma posed by Epstein (1986) (that an economy has to choose between development and equity; it cannot have both) is thereby apparently resolved. Moreover, in both classes of technologies, we obtain intergenerational streams in congruence with a view expressed by Dasgupta and Heal (1979, p. 311) and Rawls (1999, pp. 251–255) (see also Gaspart and Gosseries, 2007) that trading present consumption for future consumption is more appropriate for poorer societies, while equality considerations should dominate for richer ones.

The axiomatic underpinning of SDU is not the main focus of this paper, even though we note (in Proposition 1) that SDU satisfies all the axioms characterizing sustainable recursive SWFs, a concept analyzed in our companion paper (Asheim et al., 2009). Rather, the investigation of this paper seeks to demonstrate convincingly that SDU is an applicable criterion yielding consequences that might appeal to our ethical intuition.

Appendix

A.1. Existence of a sustainable discounted utilitarian SWF

We are given $\delta \in (0, 1)$ and $U : \mathbb{R}_+ \to \mathbb{R}$ satisfying (U.1). We want to establish existence of a function $W : \mathbf{X}_\delta \to \mathbb{R}$ satisfying (W.1)–(W.4). To this end, we first establish a basic monotonicity property, and then use that with a backward iteration device to define a function $\overline{W}$ with these properties.

Write $Z := [U(0), \infty)$. For $(a, b) \in Z \times Z$, define:

$$f(a, b) = \min\{(1 - \delta)a + \delta b, b\}. \tag{f}$$

Note that f is a well-defined function from Z^2 to Z, and furthermore:

$$f(a, b) \leq (1 - \delta)a + \delta b \quad \text{and} \quad f(a, b) \leq b \quad \text{for all } (a, b) \in Z^2. \tag{A.1}$$

Lemma A.1. *Suppose $(a, b) \in Z^2$ and $(a', b') \in Z^2$, with $(a', b') \leq (a, b)$. Then*

$$f(a', b') \leq f(a, b).$$

Further, if $b' < b$, then

$$f(a', b') < f(a, b).$$

Proof. This proof is omitted here, but included in Asheim and Mitra (2008). ■

Let ${}_0\mathbf{x} \in \mathbf{X}_\delta$ be given. For each $T \in \mathbb{N}$, define the finite sequence $\{z(0, T), \dots, z(T-1, T), z(T, T)\}$ by (1). Notice that this sequence is well-defined since $(1-\delta)\sum_{\tau=T}^{\infty} \delta^{\tau-T} U(x_\tau) \in Z$, keeping in mind that U satisfies (U.1). At each stage of the backward iteration (that is for $t = T-1, T-2, \dots, 0$) we have $z(t, T) \in Z$ by (f), since $U(x_t) \in Z$ for all $t \geq 0$.

Using Lemma A.1, we can now compare $z(0, T)$ with $z(0, T+1)$, for each $T \in \mathbb{N}$.

Lemma A.2. *For each $T \in \mathbb{N}$, we have:*

$$z(t, T) \geq z(t, T+1) \quad \text{for all } t \in \{0, \dots, T-1\}. \tag{A.2}$$

Proof. Given $T \in \mathbb{N}$, we have, from (A.1) and (1),

$$\begin{aligned} z(T, T+1) &\leq (1-\delta)U(x_T) + \delta \left[(1-\delta) \sum_{\tau=T+1}^{\infty} \delta^{\tau-T-1} U(x_\tau) \right] \\ &= (1-\delta) \sum_{\tau=T}^{\infty} \delta^{\tau-T} U(x_\tau) = z(T, T). \end{aligned}$$

Thus, applying Lemma A.1, we have:

$$z(T-1, T+1) = f(U(x_{T-1}), z(T, T+1)) \leq f(U(x_{T-1}), z(T, T)) = z(T-1, T).$$

Using Lemma A.1 repeatedly, we then obtain:

$$z(t, T+1) \leq z(t, T) \quad \text{for all } t \in \{0, \dots, T-1\}$$

which establishes (A.2). ■

With these results, we can show that $\overline{W} : \mathbf{X}_\delta \to \mathbb{R}$ defined by ($\overline{\text{W}}$) is a well-defined SDU SWF, thereby establishing existence.

Proof of Theorem 1(i). By Lemma A.2, we have $\{z(0, T)\}$ monotonically non-increasing in $T \in \mathbb{N}$, and it is bounded below by $U(0)$, so it converges. Thus, $\overline{W}$ is well-defined by ($\overline{\text{W}}$), and $\overline{W}$ maps $\mathbf{X}_\delta$ to Z since $z(0, T) \leq z(0, 1)$ for all $T \in \mathbb{N}$ and $z(0, 1) \in Z$.

By Lemma A.2, we have $\{z(t, T)\}$ monotonically non-increasing in $T > t$, and it is bounded below by $U(0)$, so it also converges. An implication of ($\overline{\text{W}}$) is that

$$\overline{W}({}_t\mathbf{x}) = \lim_{T \to \infty} z(t, T) \tag{A.3}$$

for all $t \in \mathbb{N}$.

To establish (W.1), let ${}_0\mathbf{x} \in \mathbf{X}_\delta$. We split up the analysis into three cases: (i) $U(x_0) > \overline{W}({}_1\mathbf{x})$; (ii) $U(x_0) < \overline{W}({}_1\mathbf{x})$; (iii) $U(x_0) = \overline{W}({}_1\mathbf{x})$.

In case (i), using (A.3), there is some $N \in \mathbb{N}$, such that for all $T \geq N$,

$$U(x_0) > z(1, T).$$

Thus, by (f), (1), we have $z(0, T) = z(1, T)$ for all $T \geq N$. Using ($\overline{\text{W}}$) and (A.3), we obtain $\overline{W}({}_0\mathbf{x}) = \overline{W}({}_1\mathbf{x})$, as required in (W.1).

In case (ii), using (A.3), there is some $N \in \mathbb{N}$, such that for all $T \geq N$,

$$U(x_0) < z(1, T).$$

Thus, by (f) and (1), we have $z(0, T) = (1-\delta)U(x_0) + \delta z(1, T)$ for all $T \geq N$. Using ($\overline{\text{W}}$) and (A.3), we obtain $\overline{W}({}_0\mathbf{x}) = (1-\delta)U(x_0) + \delta\overline{W}({}_1\mathbf{x})$, as required in (W.1).

In case (iii), there are two possibilities: (a) there is a subsequence of T for which $z(1, T) = U(x_0)$; (b) there is $N \in \mathbb{N}$, such that for all $T \geq N$, we have $z(1, T) \neq U(x_0)$. In case (a), using (f) and (1), we have $z(0, T) = z(1, T)$ for the subsequence of T (for which $z(1, T) = U(x_0)$). Thus, using ($\overline{\text{W}}$) and (A.3), we have $\overline{W}({}_0\mathbf{x}) = \overline{W}({}_1\mathbf{x})$. But, since $U(x_0) = \overline{W}({}_1\mathbf{x})$ in case (iii), this yields $\overline{W}({}_0\mathbf{x}) = (1-\delta)U(x_0) + \delta\overline{W}({}_1\mathbf{x})$, as required in (W.1).

In case (iii)(b), either (A) there is a subsequence of T for which $U(x_0) < z(1,T)$, or (B) there is a subsequence of T for which $U(x_0) > z(1,T)$, or both. In case (A), following the proof of case (ii), we get $\overline{W}({}_0\mathbf{x}) = (1-\delta)U(x_0)\overline{W}({}_1\mathbf{x})$, as required in (W.1). In case (B), following the proof of case (i), we get $\overline{W}({}_0\mathbf{x}) = \overline{W}({}_1\mathbf{x})$. But, since $U(x_0) = \overline{W}({}_1\mathbf{u})$ in case (iii), this yields $\overline{W}({}_0\mathbf{x}) = (1-\delta)U(x_0) + \delta\overline{W}({}_1\mathbf{x})$, as required in (W.1).

To establish (W.2), let ${}_0\mathbf{x}$ be an egalitarian stream. By (f) and (1), for each $T \in \mathbb{N}$, we have $z(t,T) = U(x_0)$ for $t \in \{0,\ldots,T-1\}$. Thus, $(\overline{\text{W}})$ implies that $\overline{W}({}_0\mathbf{x}) = U(x_0)$.

To establish (W.3), consider ${}_0\mathbf{x}', {}_0\mathbf{x}'' \in \mathbf{X}_\delta$ with ${}_0\mathbf{x}' \geq {}_0\mathbf{x}''$. We want to show that $\overline{W}({}_0\mathbf{x}') \geq \overline{W}({}_0\mathbf{x}'')$, as required in (W.3). Define in obvious notation, for each $T \in \mathbb{N}$, the finite sequences $\{z'(0,T),\ldots, z'(T-1,T), z'(T,T)\}$ and $\{z''(0,T),\ldots,z''(T-1,T), z''(T,T)\}$ as in (1). By Lemma A.1 and (1), for each $T \in \mathbb{N}$, we have $z'(t,T) \geq z''(t,T)$ for $t \in \{0,\ldots,T-1\}$. Then, by $(\overline{\text{W}})$, $\overline{W}({}_0\mathbf{x}') \geq \overline{W}({}_0\mathbf{x}'')$.

To establish (W.4), let ${}_0\mathbf{x} \in \mathbf{X}_\delta$. We want to show that $\lim_{T\to\infty}\delta^t\overline{W}({}_T\mathbf{x}) = 0$, as required in (W.4). By Lemma A.1 and (1), for each $T' \in \mathbb{N}$, we have

$$z(T,T') \leq (1-\delta)\sum_{t=T}^{\infty}\delta^{t-T}U(x_t)$$

for $T \in \{0,\ldots,T'-1\}$. Hence, by (A.3),

$$\overline{W}({}_T\mathbf{u}) = \lim_{T'\to\infty} z(T,T') \leq (1-\delta)\sum_{t=T}^{\infty}\delta^{t-T}U(x_t) \tag{A.4}$$

for $T \geq 0$. Since Z is bounded below, there does not exist $\varepsilon > 0$ and a subsequence T for which $\delta^{\mathrm{T}}\overline{W}({}_T\mathbf{x}) \leq -\varepsilon$. Suppose there exists $\varepsilon > 0$ and a subsequence T for which $\delta^{\mathrm{T}}\overline{W}({}_T\mathbf{x}) \geq \varepsilon$. By (A.4), for all T in the subsequence,

$$0 < \varepsilon \leq \delta^{\mathrm{T}}\overline{W}({}_T\mathbf{x}) \leq \delta^{\mathrm{T}}(1-\delta)\sum_{t=T}^{\infty}\delta^{t-T}U(x_t) = (1-\delta)\sum_{t=T}^{\infty}\delta^{t}U(x_t).$$

This contradicts that $\lim_{T\to\infty}(1-\delta)\sum_{t=T}^{\infty}\delta^t U(x_t) = 0$ for all ${}_0\mathbf{x} \in \mathbf{X}_\delta$. Hence, it follows that $\lim_{T\to\infty}\delta^t\overline{W}({}_0\mathbf{x}') = 0$. ■

A.2. Uniqueness of sustainable discounted utilitarian SWFs

We now study (given $\delta \in (0,1)$ and $U : \mathbb{R}_+ \to \mathbb{R}$ satisfying (U.1)) the properties of any function $W : \mathbf{X}_\delta \to \mathbb{R}$ satisfying (W.1)–(W.4).

We first state a result concerning the limit behavior of $W({}_t\mathbf{x})$ as $t \to \infty$ if the consumption stream ${}_0\mathbf{x}$ is bounded.

Lemma A.3. *If W is an SDU SWF, then, for every ${}_0\mathbf{x} \in \mathbf{X}_\varphi$,*

(i) $\lim_{t\to\infty} W({}_t\mathbf{x})$ *exists*

(ii) $\lim_{t\to\infty} W({}_t\mathbf{x}) = \lim\inf_{t\to\infty} U(x_t)$.

Proof. Since, as established in Asheim and Mitra (2008, Section A.2), any SDU SWF satisfies the axioms **O**, **M**, **IF**, **RD**, **HEF** and **RC**, this result follows from Asheim et al. (2009, Proposition 7). ■

Proof of Theorem 1(ii). Suppose there are two SDU SWFs, call them W and V, such that $W({}_0\mathbf{x}) \neq V({}_0\mathbf{x})$ for some ${}_0\mathbf{x} \in \mathbf{X}_\varphi$. Without loss of generality, let $W({}_0\mathbf{x}) > V({}_0\mathbf{x})$. If $W({}_1\mathbf{x}) \leq V({}_1\mathbf{x})$, then by Lemma A.1:

$$V({}_0\mathbf{x}) = f(U(x_0), V({}_1\mathbf{x})) \geq f(U(x_0), W({}_1\mathbf{x})) = W({}_0\mathbf{x})$$

where f is defined by (f). This is a contradiction. Thus, we must have $W({}_1\mathbf{x}) > V({}_1\mathbf{x})$, and by repeating this step we obtain:

$$W({}_t\mathbf{x}) > V({}_t\mathbf{x}) \quad \text{for all } t \geq 0. \tag{A.5}$$

We also know from Lemma A.3 that:

$$\lim_{t\to\infty} W({}_t\mathbf{x}) = \lim_{t\to\infty} V({}_t\mathbf{x}) = \lim\inf_{t\to\infty} U(x_t). \tag{A.6}$$

Thus, defining a sequence $\{k_t\}$ by $k_t = [W({}_t\mathbf{x}) - V({}_t\mathbf{x})]$ for all $t \geq 0$, we see from (A.5) and (A.6) that $k_t > 0$ for all $t \geq 0$, and $k_t \to 0$ as $t \to \infty$. It follows that there is some n for which we must have $k_{n+1} < k_n$. That is, we have:

$$0 < [W({}_{n+1}\mathbf{x}) - V({}_{n+1}\mathbf{x})] < [W({}_n\mathbf{x}) - V({}_n\mathbf{x})]. \tag{A.7}$$

We then consider three possibilities: (i) $U(x_n) \geq W({}_{n+1}\mathbf{x})$, (ii) $U(x_n) \leq V({}_{n+1}\mathbf{x})$, and (iii) $V({}_{n+1}\mathbf{x}) < U(x_n) < W({}_{n+1}\mathbf{x})$. If (i) holds, then $U(x_n) > V({}_{n+1}\mathbf{x})$, and so we have by (W.1):

$$\left.\begin{array}{ll} \text{(i)} & W({}_n\mathbf{x}) = W({}_{n+1}\mathbf{x}) \\ \text{(ii)} & V({}_n\mathbf{x}) = V({}_{n+1}\mathbf{x}) \end{array}\right\}. \tag{A.8}$$

But (A.8) clearly contradicts (A.7).

If (ii) holds, then $U(x_n) < W({}_{n+1}\mathbf{x})$, and so we have by (W.1):

$$\left.\begin{array}{ll} \text{(i)} & W({}_n\mathbf{x}) = (1-\delta)U(x_n) + W({}_{n+1}\mathbf{x}) \\ \text{(ii)} & V({}_n\mathbf{x}) = (1-\delta)U(x_n) + V({}_{n+1}\mathbf{x}) \end{array}\right\}. \tag{A.9}$$

But (A.9) implies that $[W({}_n\mathbf{x}) - V({}_n\mathbf{x})] = \delta[W({}_{n+1}\mathbf{x}) - V({}_{n+1}\mathbf{x})]$, which again contradicts (A.7).

If (iii) holds, then we have by (W.1):

$$\left.\begin{array}{ll} \text{(i)} & W({}_n\mathbf{x}) = (1-\delta)U(x_n) + W({}_{n+1}\mathbf{x}) \\ \text{(ii)} & V({}_n\mathbf{x}) = V({}_{n+1}\mathbf{x}) \end{array}\right\}. \tag{A.10}$$

By (A.10)(i) and $U(x_n) < W({}_{n+1}\mathbf{x})$, we get $W({}_n\mathbf{x}) < (1-\delta)W({}_{n+1}\mathbf{x}) + \delta W({}_{n+1}\mathbf{x}) = W({}_{n+1}\mathbf{x})$, and so by (A.10)(ii), we get $[W({}_n\mathbf{x}) - V({}_n\mathbf{x})] = [W({}_n\mathbf{x}) - V({}_{n+1}\mathbf{x})] < [W({}_{n+1}\mathbf{x}) - V({}_{n+1}\mathbf{x})]$, which again contradicts (A.7).

Since these are the only possibilities, there do not exist two SDU SWFs, W and V, such that $W({}_0\mathbf{x}) \neq V({}_0\mathbf{x})$ for some ${}_0\mathbf{x} \in \mathbf{X}_\varphi$. The result follows since, by Theorem 1(i), $\overline{W}$ is an SDU SWF. ■

A.3. Non-uniqueness of sustainable discounted utilitarian SWF

The uniqueness result of Appendix A.2 does not carry over to unbounded consumption streams. To show this, we provide another function $W : \mathbf{X}_\delta \to \mathbb{R}$ satisfying (W.1)–(W.4). Let ${}_0\mathbf{x} \in \mathbf{X}_\varphi$ be given. For each $T \in \mathbb{N}$, define the finite sequence $\{\tilde{w}(0,T), \ldots, \tilde{w}(T-1,T), \tilde{w}(T,T)\}$ as follows:

$$\left.\begin{array}{l} \tilde{w}(T,T) = \lim\inf_{t\to\infty} U(x_t) \\ \tilde{w}(T-1,T) = f(U(x_{T-1}), \tilde{w}(T,T)) \\ \ldots \\ \tilde{w}(0,T) = f(U(x_0), \tilde{w}(1,T)) \end{array}\right\}.$$

We now define $\tilde{W}({}_0\mathbf{x})$ on $\mathbf{X}_\varphi$ by

$$\tilde{W}({}_0\mathbf{x}) := \lim_{T\to\infty} \tilde{w}(0,T). \tag{$\tilde{\mathrm{W}}$}$$

Extend the domain of $\tilde{W}$ to $\mathbf{X}_\delta$ as follows. If ${}_0\mathbf{x} \in \mathbf{X}_\delta \setminus \mathbf{X}_\varphi$ has the property that $\lim\inf_{t\to\infty} U(x_t)$ exists, then the algorithm ($\tilde{\mathrm{W}}$) is still applicable. If ${}_0\mathbf{x} \in \mathbf{X}_\delta \setminus \mathbf{X}_\varphi$ does not have this property, construct each stream in the sequence $\{{}_0\mathbf{x}^n\}_{n\in\mathbb{N}}$ as follows:

$$x_t^n = \begin{cases} n & \text{if } \forall \tau \geq t, x_\tau \geq n \\ x_t & \text{if } \exists \tau \geq t \text{ s.t. } x_\tau < n, \end{cases}$$

and, since ${}_0\mathbf{x}^n \in \mathbf{X}_\varphi$ for each $n \in \mathbb{N}$, define $\tilde{W}({}_0\mathbf{x})$ in the following way:

$$\tilde{W}({}_0\mathbf{x}) := \lim_{n\to\infty} \tilde{W}({}_0\mathbf{x}^n).$$

It can be shown that $\tilde{W} : \mathbf{X}_\delta \to \mathbb{R}$ satisfies (W.1)–(W.4) and is thus an SDU SWF.

Example of non-uniqueness. Let $\delta = \frac{1}{2}$ and $U(x) = x^a$, where $\frac{1}{2} < a < 1$, implying that $U : \mathbb{R}_+ \to \mathbb{R}$ satisfies (U.1) and (U.2). Consider

$${}_0\mathbf{x} = (2^{\frac{0}{a}}, 0, 2^{\frac{1}{a}}, 0, 2^{\frac{2}{a}}, 0, 2^{\frac{3}{a}}, 0, \ldots) \in \mathbf{X}_{\frac{1}{2}},$$

leading to the utility stream ${}_0\mathbf{u} = (1, 0, 2, 0, 4, 0, 8, 0, \ldots)$. Then

$$\tilde{W}({}_0\mathbf{x}) = 0 < 1 = \overline{W}({}_0\mathbf{x}).$$

It turns out, however, that $\overline{W}$ provides an upper bound for SDU welfare, as stated in Theorem 1(iii).

Proof of Theorem 1(iii). Let ${}_0\mathbf{x} \in \mathbf{X}_\delta$. By Proposition 2 and (1), for all $T \in \mathbb{N}$, $W({}_T\mathbf{x}) \le w({}_T\mathbf{x}) = z(T, T)$. Furthermore, by (W.1) and (1), for all $t \in \{0, \ldots, T-1\}$,

$$W({}_t\mathbf{x}) = f(U(x_t), W({}_{t+1}\mathbf{x}))$$
$$w(t, T) = f(U(x_t), z(t+1, T)),$$

where f is defined by (f). By using Lemma A.1 repeatedly, we obtain:

$$W({}_0\mathbf{x}) \le z(0, T).$$

Since this holds for any $T \in \mathbb{N}$, the results follows from ($\overline{\text{W}}$). ■

A.4. An SDU SWF is a sustainable recursive SWF

In Asheim and Mitra (2008, Section A.2) we verify that any SDU SWF satisfies the axioms **O**, **M**, **IF**, **RD**, **HEF** and **RC**: Order, Monotonicity, Independent Future, Restricted Dominance, Hammond Equity for the Future, and Restricted Continuity (where axiom **IF** implies Koopmans' (1960) stationary condition). This entails that any SDU SWF is a *sustainable recursive* SWF, as defined by Asheim et al. (2009).

Here we only include the verification of axiom **RC**, which is explained below in the course of verifying it. To this end, fix $\delta \in (0, 1)$ and $U : \mathbb{R}_+ \to \mathbb{R}$ satisfying (U.1), assume that the function $W : \mathbf{X}_\delta \to \mathbb{R}$ satisfies (W.1)–(W.4) (note, however, that condition (W.4) is not needed here), and define a social welfare relation (SWR) $\succsim$ by:

$$\text{For } {}_0\mathbf{x}', {}_0\mathbf{x}'' \in \mathbf{X}_\delta,\ {}_0\mathbf{x}' \succsim {}_0\mathbf{x}'' \quad \text{if and only if} \quad W({}_0\mathbf{x}') \ge W({}_0\mathbf{x}'').$$

Let ${}_0\mathbf{x}', {}_0\mathbf{x}'' \in \mathbf{X}_\delta$ with $x'_t = x$ for all $t \ge 1$. Let ${}_0\mathbf{x}^n \in \mathbf{X}_\delta$ for $n \in \mathbb{N}$ with the property that ${}_0\mathbf{x}^n \succsim {}_0\mathbf{x}''$ for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \sup_{t\ge 0} |x^n_t - x'_t| = 0. \tag{A.11}$$

We have to show that ${}_0\mathbf{x}' \succsim {}_0\mathbf{x}''$ to verify axiom **RC**.

We first claim that $W({}_0\mathbf{x}'') \le U(x)$. Suppose, on the contrary, that $W({}_0\mathbf{x}'') > U(x)$. Then, denoting $W({}_0\mathbf{x}'')$ by ξ, we note that $\xi \in (U(x), \infty)$.

Choose $\varepsilon' > 0$ such that $U(x + \varepsilon') < \xi$. Using (A.11), we can choose $N \in \mathbb{N}$ such that $x^N_t \le x'_t + \varepsilon' = x + \varepsilon'$ for all $t \ge 1$. Then, by (W.1)–(W.3) and (A.1),

$$W({}_0\mathbf{x}'') \le W({}_0\mathbf{x}^N) \le W({}_1\mathbf{x}^N) \le U(x'_t + \varepsilon') < \xi = W({}_0\mathbf{x}'');$$

a contradiction. This establishes our claim that $W({}_0\mathbf{x}'') \le U(x)$ Thus, we have $W({}_0\mathbf{x}'') \le W({}_1\mathbf{x}')$ by (W.2).

Next, we claim that $W({}_0\mathbf{x}'') \leq W({}_0\mathbf{x}')$. Suppose, on the contrary that $\eta := [W({}_0\mathbf{x}'') - W({}_0\mathbf{x}')] > 0$. Then, by (W.2) and (W.3), we have

$$U(0) \leq W({}_0\mathbf{x}') < W({}_0\mathbf{x}'') \leq U(x)$$

so that $U(x) - U(0) \geq \eta > 0$. Using (A.11), we can choose $N \in \mathbb{N}$ so that $\bar{x}^N := \sup_{t\geq 1} x_t^N$ and $\underline{x}^N := \inf_{t\geq 1} x_t^N$ exist and

$$|U(x_0^N) - U(x_0')| < \eta, \qquad U(\bar{x}^N) < U(x) + \eta, \qquad U(\underline{x}^N) > U(x) - \eta. \tag{A.12}$$

Note that it follows from (A.1) that, whenever $(a, b) \in Z^2$ and $(a', b') \in Z^2$ satisfy $|a' - a| < \eta$ and $|b' - b| < \eta$, we must have

$$|f(a', b') - f(a, b)| < \eta. \tag{A.13}$$

We now show that:

$$|W({}_0\mathbf{x}^N) - W({}_0\mathbf{x}')| < \eta. \tag{A.14}$$

Note that by (A.12), $W({}_1\mathbf{x}^N) \leq U(\bar{x}^N) < U(x) + \eta = W({}_1\mathbf{x}') + \eta$, using (W.2) and (W.3). Similarly, $W({}_1\mathbf{x}^N) \geq U(\underline{x}^N) > U(x) - \eta = W({}_1\mathbf{x}') - \eta$. Thus,

$$|W({}_1\mathbf{x}^N) - W({}_1\mathbf{x}')| < \eta. \tag{A.15}$$

We have $W({}_0\mathbf{x}^N) = f(x_0^N, W({}_1\mathbf{x}^N))$ and $W({}_0\mathbf{x}') = f(x_0', W({}_1\mathbf{x}'))$. Thus, using (A.12), (A.13) and (A.15), we obtain (A.14).

In particular, (A.14) implies that:

$$W({}_0\mathbf{x}') + \eta = W({}_0\mathbf{x}'') \leq W({}_0\mathbf{x}^N) < W({}_0\mathbf{x}') + \eta;$$

a contradiction. This establishes the claim that $W({}_0\mathbf{x}') \geq W({}_0\mathbf{x}'')$ and so ${}_0\mathbf{x}' \succsim {}_0\mathbf{x}''$.

The same kind of argument can be used to show ${}_0\mathbf{x}' \precsim {}_0\mathbf{x}''$ if ${}_0\mathbf{x}^n \precsim {}_0\mathbf{x}''$ for all $n \in \mathbb{N}$.

A.5. Proofs of Lemmas 1–6

Proof of Lemma 1. Let $y_0 > 0$ and $\delta \in (0, 1)$ be given, implying that $(1+\delta)/2\delta > 1$. While $f(k)/k > 1$ for all $k > 0$, we have $\lim_{k\to\infty}[f(k)/k] = 1$. Thus, there is $K > y_0$ such that $f(k)/k \leq (1+\delta)/2\delta$ for all $k \geq K$. This implies that, for all $k \geq K$, we have $\delta f(k)/k \leq (1+\delta)/2 \equiv \mu < 1$.

Define $k_0 = K$, and $k_{t+1} = f(k_t)$ for $t \geq 0$, $a_t = f(k_t)/k_t$ for $t \geq 0$, and $\pi_t = \prod_{s=0}^{t} a_s$ for $t \geq 0$. Then, for every y_0-feasible stream, we have $x_{t+1} \leq y_{t+1} \leq f(y_t - x_t) \leq f(y_t) \leq f(k_t) = a(t)k(t) = \pi(t)K$, and so:

$$\delta^{t+1}x_{t+1} \leq \delta^{t+1}\pi_t K \leq \mu^{t+1}K \quad \text{for all } t \geq 0.$$

Hence, for every y_0-feasible stream, $\sum_{t=0}^{\infty} \delta^t x_t \leq K/(1-\mu) < \infty$. ■

Proof of Lemma 2. Let $(y_0, m_0) \gg 0$ and $\delta \in (0, 1)$ be given. Define $f(k) = F(k, m_0, 1) + k$ for $k \geq 0$. Then $f(k)/k > 1$ for all $k > 0$, while we have $\lim_{k\to\infty}[f(k)/k] = 1$. Therefore, the argument given in the proof of Lemma 1 applies here as well. ■

Proof of Lemma 3. The existence of an efficient and egalitarian (y_0, m_0)-feasible program $({}_0\mathbf{y}^e, {}_0\mathbf{m}^e, {}_0\mathbf{k}^e)$, such that the associated (y_0, m_0)-feasible consumption stream ${}_0\mathbf{x}^e \gg 0$, follows from Dasgupta and Mitra (1983, Proposition 5); uniqueness follows from (F.1). Property (24) of the price sequence ${}_0\mathbf{p}$ follows from Asheim (1988, Lemma 3 and Proposition 1). Property (25) of maximization of the present value of the consumption stream at ${}_0\mathbf{x}^e$ follows from Dasgupta and Mitra (1983, Theorem 1). ■

Proof of Lemma 4. The price sequence ${}_0\mathbf{p} \gg 0$, determined by (23), and supporting the unique (y_0, m_0)-feasible program $({}_0\mathbf{y}^e, {}_0\mathbf{m}^e, {}_0\mathbf{k}^e)$ obtained in Lemma 3, satisfies (24). Denote (p_1/p_0) by ρ. Then, by using (24), we have $\theta > 0$, such that $(p_{t+1}/p_t) > \rho + \theta$ for all $t \geq 1$. Let $T \geq 2$ be given. Then, for $t \in \{1, \ldots, T\}$, we have

$$p_{t+1} > \rho p_t + \theta p_t. \tag{A.16}$$

Adding up the inequalities in (A.16) from $t = 1$ to $t = T$, we get:

$$p_2 + p_3 + \cdots + p_{T+1} > \rho(p_1 + p_2 + \cdots + p_T) + \theta p_1. \tag{A.17}$$

Adding the trivial equality $p_1 = \rho p_0$ to (A.17), we obtain:

$$p_1 + p_2 + p_3 + \cdots + p_{T+1} > \rho(p_0 + p_1 + p_2 + \cdots + p_T) + \theta p_1.$$

This yields

$$\left[\frac{p_1 + p_2 + p_3 + \cdots + p_{T+1}}{p_0 + p_1 + p_2 + \cdots + p_T}\right] > \rho + \left[\frac{\theta p_1}{p_0 + p_1 + p_2 + \cdots + p_T}\right] \geq \rho + \left[\frac{\theta p_1}{\sigma}\right], \tag{A.18}$$

where $\sigma = \sum_{t=0}^{\infty} p_t$. Letting $T \to \infty$ in (A.18), we get:

$$\delta^{\infty}(y_0, m_0) \geq \rho + \left[\frac{\theta p_1}{\sigma}\right] > \rho = \delta^0(y_0, m_0),$$

which is the desired result. ■

Proof of Lemma 5. Assume that $({}_0\mathbf{y}, {}_0\mathbf{m}, {}_0\mathbf{k})$ is a (y_0, m_0)-feasible program where the associated (y_0, m_0)-feasible stream ${}_0\mathbf{x} \gg 0$ satisfies $\liminf_{T\to\infty} w({}_T\mathbf{x}) > U(0)$. By (G.1) and (G.3), there exists $\tilde{k} \geq 1$ satisfying $F(1, m_0/\tilde{k}) \leq 1/\delta$. Note that $k_T \to \infty$ as $T \to \infty$ and $m_t > 0$ for all $t \geq 0$ (since otherwise $\liminf_{T\to\infty} w({}_T\mathbf{x}) = U(0)$, contradicting the hypothesis of the lemma). Choose a time τ such that $k_\tau \geq \tilde{k} \geq 1$. Consider the efficient and egalitarian (y_τ, m_τ)-feasible program $({}_\tau\mathbf{y}^e, {}_\tau\mathbf{m}^e, {}_\tau\mathbf{k}^e)$, with supporting price sequence ${}_\tau\mathbf{p}$. By Lemma 3 and (G.1)–(G.3),

$$\left[\frac{1}{\delta^0(y_\tau, m_\tau)}\right] = \left[\frac{p_\tau}{p_{\tau+1}}\right] = F_k(k_\tau, m_\tau - m^e_{\tau+1}) \leq F(k_\tau, m_\tau - m^e_{\tau+1})/k_\tau$$
$$\leq F(1, (m_\tau - m^e_{\tau+1})/k_\tau) < F(1, m_0/\tilde{k}) \leq \frac{1}{\delta},$$

thereby establishing that there is a finite time τ such that $\delta^0(y_\tau, m_\tau) \geq \delta$. ■

Proof of Lemma 6. Existence follows from Asheim (1988, Proposition 2, sufficiency part). Parts (i) and (ii) follow from Asheim (1988, Lemma 4(a) and (c)). That ${}_0\mathbf{x}^*$ is egalitarian if $\delta^{\infty}(y_0, m_0) \geq \delta$ follows from Asheim (1988, Lemma 4(b)). The proof of Asheim (1988, Lemma 4) implies the two-phase structure of ${}_0\mathbf{x}^*$, stated in part (ii). Finally, Lemma 5 of this paper establishes that τ of part (ii) is finite. ■

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